

EQUIVARIANT COHOMOLOGY OF $(\mathbb{Z}_2)^r$ -MANIFOLDS AND SYZYGIES

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ABSTRACT. We consider closed manifolds with $(\mathbb{Z}_2)^r$ -action, which are obtained as intersections of products of spheres of a fixed dimension with certain ‘generic’ hyperplanes. This class contains the real versions of the ‘big polygon spaces’ defined and considered by M.Franz in [12]. We calculate the equivariant cohomology with \mathbb{F}_2 -coefficients, which in many examples turns out to be torsion-free but not free and realizes all orders of syzygies, which are in concordance with the restrictions proved in [4]. The final results for the real versions are analogous to those for the big polynomial spaces in [12], where $(S^1)^r$ -actions and rational coefficients are considered, but we consider also a wider class of manifolds here and the point of view as well as the method of proof, for which it is essential to consider equivariant cohomology for divers - but related - groups, are quite different.

1. INTRODUCTION

In the papers [2] and [3] the equivariant cohomology (with coefficients in a field of characteristic 0) of spaces equipped with an action of a torus $T = (S^1)^r$ was studied, in particular the relation between the so-called Atiyah–Bredon sequence and the notion of syzygies coming from commutative algebra. Among the results is the following theorem (see [2], Cor. 1.4):

Theorem 1.1. *Let X be a compact orientable T -manifold. If $H_T^*(X)$ is a syzygy of order $k \geq r/2$, then it is free over $H^*(BT)$.*

In [12] examples of $T = (S^1)^r$ -manifolds were given, which show that the restriction on the order of syzygies obtained in [2], are sharp. Coefficients were taken in \mathbb{Q} .

In this note we consider actions of a 2-torus $G = (\mathbb{Z}_2)^r$ and coefficients in the field \mathbb{F}_2 of characteristic 2. All major analogous results of [2] and [3], in particular Theorem 1.1 above, turn out to be true in this setting. Nevertheless, some of them require new methods of proof, basically because, in contrast to $T = (S^1)^r$, $G = (\mathbb{Z}_2)^r$ has only finitely many subgroups and because the field \mathbb{F}_2 has only finitely many elements. This is carried out in a so far unpublished manuscript [4], even for the not quite analogous case of $G = (\mathbb{Z}_p)^r$ -actions and coefficients in a field k of characteristic $p > 0$, p an odd prime, which is somewhat more involved than the $p = 2$ case.

The results in this note for the real versions of the ‘big polygon spaces’ with $(\mathbb{Z}_2)^r$ -actions are in a sense analogous to those for $(S^1)^r$ -actions in [12], i.e. among other results we show (see Cor. 3.16, Remark 3.17, and compare [12], Cor. 5.3 for the case of $(S^1)^r$ -actions):

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Theorem 1.2. *Let k and r be integers with $k < r/2$, then there exists a compact $(\mathbb{Z}_2)^r$ -manifold, N_0 , such that $H_{(\mathbb{Z}_2)^r}^*(N_0; \mathbb{F}_2)$ is a k -th syzygy over $H^*(B(\mathbb{Z}_2)^r; \mathbb{F}_2)$ but not a $(k+1)$ -th syzygy.*

Compared to [12] we take a different point of view and the proofs are also quite different. We consider the equivariant cohomology for divers groups acting on a class of manifolds, which contains the real analogues of the ‘big polygon spaces’ of M.Franz, but also the more general ‘big chain spaces’ (see Remark 3.1, (3) and (4), Remark 3.10 and Cor. 3.13) which are not considered in [12]. Certain familiarity with equivariant cohomology and P.A. Smith-Theory is assumed throughout. Standard references are e.g. [6], [7], [5].

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2. SOME BASIC DEFINITIONS AND FUNDAMENTAL RESULTS

We will not cite here all analogous results to those in [2] and [3] for the case of 2-tori and \mathbb{F}_2 -coefficients but concentrate on the result which is relevant in view of the later examples.

Let $G = (\mathbb{Z}_2)^r$ be a 2-torus and $R = H^*(BG) = \mathbb{F}_2[x_1, \dots, x_r]$ the polynomial ring in the variables x_1, \dots, x_r of degree 1.

We recall the notion of syzygies from commutative algebra, see e.g. [8]. A finitely generated R -module M is called a j -th syzygy if there is an exact sequence

$$(2.1) \quad 0 \rightarrow M \rightarrow F^1 \rightarrow \dots \rightarrow F^j$$

with finitely generated free R -modules F^1, \dots, F^j . The first syzygies are exactly the torsion-free R -modules, and the j -th syzygies with $j \geq r$ are the free modules.

An easy way to obtain syzygies over the polynomial ring R is to use the Koszul resolution

$$(2.2) \quad 0 \longrightarrow R \xrightarrow{\delta_r} R^{\binom{r}{r-1}} \longrightarrow \dots \longrightarrow R^{\binom{r}{1}} \xrightarrow{\delta_1} R \xrightarrow{\delta_0} \mathbb{F}_2 \longrightarrow 0,$$

indeed, the image of δ_j , K_j , is obviously a j -th syzygy by definition, but it is not a $(j+1)$ -th syzygy, because the homological dimension over R , $hdim_R(M)$, of a $(j+1)$ -th syzygy M over R is at most $r - (j+1)$, while $hdim_R(K_j) = r - j$.

In [1, Prop.] Allday proves for rational coefficients the following result for a suitable Poincaré duality (PD – for short) space X on which $T = (S^1)^r$ acts, e.g. a compact orientable T -manifold. If $H_T^*(X)$ has homological dimension 1, then it has $H^*(BT)$ -torsion. In particular, if $r = 2$, i.e., if $T = S^1 \times S^1$, then $H_T^*(X)$ is torsion-free if and only if it is free.

Analogous results hold for $(\mathbb{Z}_2)^r$ instead of T , and \mathbb{F}_2 -coefficients. But the above equivalence breaks down for $r > 2$; see [13] for counterexamples. The correct generalization of Allday’s result is as follows.

Proposition 2.1. *Let X be a PD -space with a G -action, which is a compact G -CW complex, e.g. a compact orientable G -manifold.*

If $H_G^(X)$ is a k -th syzygy for some $k \geq r/2$, then it is free over $H^*(BG)$.*

Proof. Compare [2], Proposition 5.12(2) for the result in case of $G = (S^1)^r$ -actions and rational coefficients. A proof for the case $G = (\mathbb{Z}_2)^r$ and \mathbb{F}_2 -coefficients is contained in [4]. \square

While in [2] it was shown by examples that actually all orders of syzygies can occur as equivariant cohomology modules of non-compact G -manifolds; examples of compact G -manifolds or PD -spaces which realize all orders of syzygies $< r/2$ were not given there. This is done in [12] for the case of $(S^1)^r$ -actions. Here we give similar results, but essentially different proofs for G -actions, where $G = (\mathbb{Z}_2)^r$. From now on we always take \mathbb{F}_2 -coefficients. A G -space is *equivariantly formal* in the sense of [14] if and only if the equivariant cohomology, $H_G^*(X)$ is isomorphic to $H^*(X) \otimes R$ as an R -module (but not necessarily as an R -algebra). We denote this property by *CEF* (*cohomologically equivariantly formal*) to distinguish it from notions of formality in rational homotopy theory.

The following Mayer-Vietoris type theorem is basic for our calculations.

Theorem 2.2. *Let M be a G -manifold, G a 2-torus. Assume that $M^+, M^- \subset M$ are G -invariant, with $M = M^+ \cup M^-$, such that M, M^+, M^- are CEF, and the action on $M^0 := M^+ \cap M^-$ is fixed point free. Assume also that the maps $H^*(M) \rightarrow H^*(M^\pm)$ induced by the inclusions are surjective. Then one has the following Mayer-Vietoris diagram:*

$$(2.3) \quad \begin{array}{ccc} H_G^*(M) & \longrightarrow & H_G^*(M^+) \\ \downarrow & & \downarrow \\ H_G^*(M^-) & \longrightarrow & H_G^*(M^0) \end{array}$$

All maps in the above diagram are surjective and the long exact Mayer-Vietoris sequence decomposes into short exact sequences

$$(2.4) \quad 0 \longrightarrow H_G^*(M) \xrightarrow{(\xi^+, \xi^-)} H_G^*(M^+) \oplus H_G^*(M^-) \longrightarrow H_G^*(M^0) \longrightarrow 0$$

one has

$$(2.5) \quad H_G^*(M^0) \cong H_G^*(M) / (\ker \xi^+ \oplus \ker \xi^-) \cong (H_G^*(M^+) \oplus H_G^*(M^-)) / (\xi^+, \xi^-) H_G^*(M).$$

Proof. Since M, M^\pm are CEF, one has $H_G^*(M) \cong H^*(M) \otimes R$ and $H_G^*(M^\pm) \cong H^*(M^\pm) \otimes R$ as R -modules. Although the maps in equivariant cohomology might not be the canonical extensions of the corresponding maps in non-equivariant cohomology, still the former are surjective because the latter are so by assumption (see e.g. [5] Lemma (A.7.3)(2)). On the other hand, because the fixed point set $(M^0)^G$ is empty, one has

$$(2.6) \quad H_G^*(M^G) \cong H_G^*((M^+)^G) \oplus H_G^*((M^-)^G).$$

Since the inclusions of the fixed point set of M induces an injection in equivariant cohomology, the map $(\xi^+, \xi^-) : H_G^*(M) \rightarrow H_G^*(M^+) \oplus H_G^*(M^-)$ is also injective, for the composition with $H_G^*(M^+) \oplus H_G^*(M^-) \rightarrow H_G^*((M^+)^G) \oplus H_G^*((M^-)^G)$ coincides with the injective map

$$(2.7) \quad H_G^*(M) \rightarrow H_G^*(M^G) \cong H_G^*((M^+)^G) \oplus H_G^*((M^-)^G).$$

This means that the long exact Mayer-Vietoris sequence decomposes into short exact sequences and the above Mayer-Vietoris diagram is cocartesian. Hence together with the maps on the upper and left side of the diagram also those on the lower and right side are surjective. \square

Remark 2.3. Under the conditions of Theorem 2.2 the sequence (2.4) is a free resolution of the R -module $H_G^*(M^0)$ since $H_G^*(M)$ and $H_G^*(M^\pm)$ are free R -modules. Since ξ^\pm are surjective, $\ker \xi^\pm$ are also free, and $\ker \xi^+ \cap \ker \xi^- = 0$ since (ξ^+, ξ^-) is injective. Hence the following sequence is also a free resolution of $H_G^*(M^0)$:

$$(2.8) \quad 0 \longrightarrow \ker \xi^- \longrightarrow H_G^*(M)/\ker \xi^+ \longrightarrow H_G^*(M^0) \longrightarrow 0$$

Here we identify $\ker \xi^-$ with its isomorphic image in $H_G^*(M)/\ker \xi^+$.

3. THE MANIFOLDS

The manifolds we consider in this section are intersections of products of spheres of a fixed dimension with a number of hyperplanes of a particular type. The actions are just the restrictions of the canonical action on the ambient Euclidean space to some of the coordinates. Among these manifolds are the real analogues of the ‘big polygon spaces’ considered in [12] (cf. Remark 3.1.(1) below). We calculate the equivariant cohomology with respect to different groups. It turns out that the equivariant cohomology for these manifolds with respect to certain subgroups is often torsion-free but not free and realizes all orders of syzygies which are in concordance with Proposition 2.1.

Let $S^{m+n-1} := \{(x_1, \dots, x_m, y_1, \dots, y_n) \in \mathbb{R}^{m+n}; \sum_{i=1}^m x_i^2 + \sum_{j=1}^n y_j^2 = 1\}$, $M := (S^{m+n-1})^r$ for $m \geq 2, n \geq 0, r \geq 1$. A point $w \in M$ is given by the coordinates $w = ((x_{1,1}, \dots, x_{m,1}, y_{1,1}, \dots, y_{n,1}), (x_{1,2}, \dots, x_{m,2}, y_{1,2}, \dots, y_{n,2}), \dots, (x_{1,r}, \dots, x_{m,r}, y_{1,r}, \dots, y_{n,r}))$. Let $\ell := (l_1, \dots, l_r) \in (\mathbb{R} \setminus \{0\})^r$. The vector ℓ is called *generic* if and only if $\sum_{j=1}^r l_j \epsilon_j \neq 0$ for all $\epsilon_j = \pm 1$. We assume throughout this note that ℓ is generic. We define $f_i : M \rightarrow \mathbb{R}$ by $f_i(w) := \sum_{j=1}^r l_j x_{i,j}$ for $i = 1, \dots, m$. We actually assume that all l_j are positive, since one can replace the coordinate $x_{i,j}$ by $-x_{i,j}$, if necessary. Set $M_0 := M$, and for $i = 1, \dots, m$, $M_i := \{w \in M; f_\mu(w) = 0 \text{ for } \mu = 1, \dots, i\}$; furthermore let $g_i := f_i|_{M_{i-1}}$, so $g_i : M_{i-1} \rightarrow \mathbb{R}$. We put $N_c := g_m^{-1}(c)$, for $c \in \mathbb{R}$. Since N_c is homeomorphic to N_{-c} (by multiplying the coordinates $(x_{m,1}, \dots, x_{m,r})$ with -1), we may assume $c \geq 0$.

Remark 3.1. Spaces of the above type have been considered by many mathematicians in different contexts, see [9], [10], [11], [12], [15] and the references therein, e.g.:

- (1) Big polygon spaces: These spaces are studied in [12]. They are the complex version of the above spaces $N_0 = M_m$ with $\ell = (l_1, \dots, l_r) \in \mathbb{R}^r$ and $n \geq 1$ considered with an action of $(S^1)^r$. (It is shown in [12] that without loss of generality one may assume that $\ell \in (\mathbb{R} \setminus \{0\})^r$.)
- (2) Polygon spaces: If $\ell = (l_1, \dots, l_r) \in \mathbb{R}_{>0}^r$, the space $\{w \in M_m; y_{\nu,1} = \dots = y_{\nu,r} = 0 \text{ for } \nu = 1, \dots, n\}$ (which amounts to the same as just taking $n = 0$) is homeomorphic to the space of polygons (resp. the free polygon space) $\tilde{\mathcal{N}}_m^r(\ell)$ in \mathbb{R}^m for the length vector ℓ (cf. [15], Chapt.10.3; p.445). These spaces (named $E_m(\ell)$ in [11]) and in particular their non-equivariant cohomology with \mathbb{Z}_2 -coefficients are studied in [10] and [11].

- (3) Big chain space: With the notation as in (2), the space $\{w \in f_m^{-1}(c); y_{\nu,1} = \dots = y_{\nu,r} = 0 \text{ for } \nu = 1, \dots, n\}$ is homeomorphic to the big chain space $\mathcal{BC}_m^{r+1}(\tilde{\ell})$ for $\tilde{\ell} = (l_1, \dots, l_r, c)$, in [15], Chapt.10.3; p.444. We also consider N_c for $n \geq 1$ and $c \neq 0$ as a big chain space.
- (4) Chain spaces: Again with the notation as in (2), the space $\{w \in g_m^{-1}(c); y_{\nu,1} = \dots = y_{\nu,r} = 0 \text{ for } \nu = 1, \dots, n\}$ is homeomorphic to the chain space $\mathcal{C}_m^{r+1}(\tilde{\ell})$ for $\tilde{\ell} = (l_1, \dots, l_r, c)$, in [15], Chapt.10.3; p.444. They were also studied in [9].

Our aim is to calculate the equivariant cohomology of the spaces M_i , which are just intersections of M with certain hyperplanes, with respect to the standard linear $G = (\mathbb{Z}_2)^r$ -action on all coordinates $y_{\nu,1}, \dots, y_{\nu,r}$ for $\nu = 1, \dots, n$. So the spaces described in (2) and (4) of the above remark occur as the fixed point sets of these actions on the corresponding spaces, where $y_{\nu,1}, \dots, y_{\nu,r}$ for $\nu = 1, \dots, n$ are arbitrary.

Proposition 3.2. (1) For $i = 1, \dots, m$ the functions $f_i : M \rightarrow \mathbb{R}$ are Morse functions with the set of isolated critical points $C_i := \{w \in M; \text{all coordinates equal to zero, except } x_{i,j} = \pm 1 \text{ for } j = 1, \dots, r\}$.

(2) For $i = 1, \dots, m$ one has: If ℓ is generic, then M_{i-1} is a closed manifold, and $g_i : M_{i-1} \rightarrow \mathbb{R}$ is a Morse function with isolated critical points C_i as above.

Proof. The proof of part (1) is essentially contained in [15], Lemma 10.3.1. We give some details and introduce notation in view of the proof of part (b).

Part (1): The map $S^{m+n-1} \rightarrow \mathbb{R}$ given by $(x_{1,j}, \dots, x_{m,j}, y_{1,j}, \dots, y_{n,j}) \mapsto l_j x_{i,j}$ for a single sphere is clearly a Morse function with two non-degenerated critical points, given by $x_{i,j} = \pm 1$ and all other coordinates equal to 0. The corresponding Hesse matrix is a diagonal matrix with entries $\pm l_{i,j}$ along the diagonal, if one uses the coordinate system for the sphere around the critical points obtained by projecting to the coordinates different from $x_{i,j}$. It follows that for the map f_i on $(S^{m+n-1})^r$ the set of critical points is just C_i above, and the Hesse matrix with respect to the coordinate systems chosen above is a huge diagonal matrix, which after arranging the variables in lexicographical order (i.e.: $w = (x_{1,1}, x_{1,2}, \dots, x_{1,r}, x_{2,1}, \dots, x_{2,r}, \dots, x_{m,r}, y_{1,1}, \dots, y_{n,r})$, and omitting the coordinates $x_{i,j}$ for $j = 1, \dots, r$), can be view as consisting of $m+n-1$ diagonal blocks of the form

$$(3.1) \quad \begin{pmatrix} \pm l_1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & \pm l_2 & 0 & \cdot & \cdot & \cdot \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ 0 & 0 & 0 & \cdot & \cdot & \pm l_r \end{pmatrix}$$

In particular, all critical points in C_i are regular, so f_i is a Morse function, and $(\mathbb{R} \setminus f_i(C_i))$ is the set of regular values.

Part (2): Proof by induction on i . The beginning, $i = 1$, is clear by part (a).

By induction hypothesis $g_{i-1} : M_{i-2} \rightarrow \mathbb{R}$ is a Morse function on the compact manifold M_{i-2} with isolated critical points C_{i-1} . Since ℓ is generic, g_{i-1} is non zero on C_{i-1} . So $0 \in \mathbb{R}$ is a regular value of g_{i-1} and hence $g_{i-1}^{-1}(0) = M_{i-1}$ is a compact manifold. Since C_i above is contained in M_{i-1} , the points in C_i are also critical points of g_i . We first show that there are no other critical points of g_i . For

a critical point $w \in M_{i-1}$ of g_i the differential dg_i must be normal to M_{i-1} . This gives the following conditions for the single coordinates :

- (1) $l_j = c_j x_{i,j}$ for $j = 1, \dots, r$ and some $c_j \neq 0$
- (2) $x_{\mu,j} = 0$ for $\mu > i; j = 1, \dots, r$, and $y_{\nu,j} = 0$ for $\nu = 1, \dots, n; j = 1, \dots, r$
- (3) $x_{\mu,j}/x_{i,j} = c_\mu$ for $\mu < i; j = 1, \dots, r$

On the other hand one has

$$(3.2) \quad \sum_{\mu=1}^m x_{\mu,j}^2 + \sum_{\nu=1}^n y_{\nu,j}^2 = \sum_{\mu=1}^i x_{\mu,j}^2 = x_{i,j}^2 + \sum_{\mu=1}^{i-1} c_\mu x_{i,j}^2 = 1$$

Hence $x_{i,j}^2 = (1 + \sum_{\mu=1}^{i-1} c_\mu^2)^{-1}$ is independent of j . Similarly for $\mu < i$ one has $x_{\mu,j}^2 = 1 - \sum_{\xi \neq \mu} x_{\xi,j}^2 = 1 - \sum_{\xi \neq \mu} c_\xi^2 x_{i,j}^2$ is also independent of j . Since l is generic $\sum_{j=1}^r l_j x_{\mu,j} = 0$ for $\mu = 1, \dots, i-1$ implies that $x_{\mu,j} = 0$ for $\mu = 1, \dots, i-1; j = 1, \dots, r$. All together one has $x_{i,j}^2 = 1$ for $j = 1, \dots, r$ and $x_{\mu,j} = 0$ for $j = 1, \dots, r$ and $\mu \neq i$. Also $y_{\nu,j} = 0$ for $\nu = 1, \dots, n; j = 1, \dots, r$. This means that the set of singular points of g_i is C_i .

We show next that the critical points are regular. This amounts to proving that the diagonal Hesse form of part (a) is still regular when restricted to the intersection with the linear subspaces of $(\mathbb{R}^{m+n})^r$ given by the equations $\sum_{j=1}^r l_j x_{\mu,j} = 0$ for $\mu = 1, \dots, i-1$. Without restriction we may assume that $l_r = 1$. Hence for $\mu = 1, \dots, i-1$ we have $x_{\mu,r} = -\sum_{j=1}^{r-1} l_j x_{\mu,j}$. For the coordinates $(x_{1,1} \dots x_{1,r-1}) \dots (x_{i-1,1} \dots x_{i-1,r-1}), (x_{i,1} \dots x_{i,r}) \dots (x_{m,1} \dots x_{m,r}), (y_{1,1} \dots y_{n,r})$ we get the same blocks for the Hesse matrix as in part (a), except for those where the coordinate $x_{\mu,r}$ is deleted. In the latter case the blocks look as follows:

$$(3.3) \quad \begin{pmatrix} \pm l_1 + l_1^2 & l_1 l_2 & . & . & . & l_1 l_{r-1} \\ l_1 l_2 & \pm l_2 + l_2^2 & . & . & . & l_2 l_{r-1} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ l_1 l_{r-1} & . & . & . & . & \pm l_{r-1} + l_{r-1}^2 \end{pmatrix}$$

We have to show that the determinant of such a block matrix is non-zero. Adding appropriate multiples of the last column to the first $(r-2)$ columns gives the following matrix:

$$(3.4) \quad \begin{pmatrix} \pm l_1 & 0 & . & . & . & l_1 l_{r-1} \\ 0 & \pm l_2 & . & . & . & l_2 l_{r-1} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \mp l_1 l_{r-1} & \mp l_2 l_{r-1} & . & . & . & \pm l_{r-1} + l_{r-1}^2 \end{pmatrix}$$

Adding (± 1) times the first $(r-2)$ rows to the last row gives a triangular matrix with determinant $(\mp l_1)(\mp l_2) \dots (\mp l_{r-2})(l_{r-1})(\pm 1 \mp l_1 \mp l_2 \dots \mp l_{r-2} \pm l_{r-1})$. This term is non-zero since $l_j \neq 0$ for $j = 1, \dots, r$ as is $(\pm 1 \mp l_1 \mp l_2 \dots \mp l_{r-2} \pm l_{r-1})$, because l is generic. \square

The Morse inequalities give the following result:

Corollary 3.3. *For $i = 1, \dots, m$, $\dim_{\mathbb{F}_2} H^*(M_{i-1}) \leq 2^r$.*

The set C_i can also be viewed as the fixed point set of the involution on M_{i-1} , which is given by multiplying all coordinates of $m \in M_{i-1}$ with ± 1 , except for the $x_{i,j}, j = 1, \dots, r$. By the P.A. Smith inequalities one has $\dim_{\mathbb{F}_2} H^*(M_{i-1}) \geq 2^r$.

Corollary 3.4. *The maps $g_i : M_{i-1} \rightarrow \mathbb{R}$ are perfect Morse functions for $i = 1, \dots, m$ and coefficients \mathbb{F}_2 , and $\dim_{\mathbb{F}_2} H^*(M_{i-1}) = 2^r$.*

If a 2-torus acts on M_{i-1} with fixed point set C_i , then this action is *CEF*. Furthermore: If g_i is equivariant with respect to the 2-torus action and the trivial action on \mathbb{R} then for a regular value $c \in \mathbb{R}$ of g_i the subspaces $g^{-1}(-\infty, c) \simeq g^{-1}(-\infty, c]$ and $g^{-1}(c, \infty) \simeq g^{-1}[c, \infty)$ are also *CEF*. If the equivariant cohomology of these spaces is known for a regular value $c \in \mathbb{R}$ of g_i , then one can calculate the equivariant cohomology of $g^{-1}(c)$ by a Mayer-Vietoris argument (see Theorem 2.2). We apply this below. Let the action of $\tilde{G}_i = \mathbb{Z}_2 \times (\mathbb{Z}_2)^r$ on M be defined by the standard linear presentation of $(\mathbb{Z}_2)^r$ on the coordinates $y_{\nu,1}, \dots, y_{\nu,r}$ and the ‘diagonal’ involution which multiplies the coordinates $x_{\mu,\nu}$ by ± 1 , except for the coordinates $x_{i,j}, j = 1, \dots, r$ where the action is trivial. Note that $C_i = M^{\tilde{G}_i}$ is contained in M_{i-1} , and M_{i-1} is \tilde{G}_i -invariant. So the \tilde{G}_i -action on M_{i-1} is *CEF* for $i = 1, \dots, m$.

Although we are mainly interested in calculating the equivariant cohomology of the manifolds M_i with respect to the action of G above, it turns out to be useful to calculate the equivariant cohomology with respect to the action of the bigger group \tilde{G}_i first.

Proposition 3.5. $H_{\tilde{G}_i}^*(M) \cong \mathbb{F}_2[s_1, \dots, s_r, t, t_1, \dots, t_r] / \{s_j \bar{s}_j, j = 1, \dots, r\}$, where $\bar{s}_j = s_j + t^{m-1} t_j^n, (|s_j| = m + n - 1, |t_j| = |t| = 1)$.

Proof. The proof follows by induction on r using the Künneth Theorem (cf.[15], Proposition 10.3.5, with different notation). \square

For $J = \{j_1, \dots, j_k\} \subset \{1, \dots, r\}$ we denote by $p_{i,J}$ the fixed point in $C_i \subset M$ with
$$x_{i,j} = \begin{cases} +1 & \text{for } j \in J \\ -1 & \text{for } j \notin J \end{cases}$$

In Proposition 3.5 we can choose the variables s_1, \dots, s_r in such a way that the restriction to a fixed point $p_{i,J}$ in equivariant cohomology is given by the following proposition.

Proposition 3.6. *The inclusion $p_{i,J} \in M$ induces the following map in equivariant cohomology*

$$H_{\tilde{G}_i}^*(M) \cong \mathbb{F}_2[s_1, \dots, s_r, t, t_1, \dots, t_r] / \{s_j \bar{s}_j, j = 1, \dots, r\} \longrightarrow H_{\tilde{G}_i}^*(p_{i,J}) \cong \mathbb{F}_2[t, t_1, \dots, t_r],$$

$$s_j \longmapsto \begin{cases} t^{m-1} t_j^n & \text{if } j \in J \\ 0 & \text{if } j \notin J \end{cases}$$

Proof. Again the proof can be given by induction on r using the Künneth Theorem. \square

Our main aim is to calculate the equivariant cohomology of $N_0 = M_m$. Before we get to the equivariant cohomology of N_0 with respect to the $G = (\mathbb{Z}_2)^r$ -action we consider the equivariant cohomology with respect to an extended action of the group $\tilde{G} = \mathbb{Z}_2 \times G$ where the first factor \mathbb{Z}_2 acts by multiplication with ± 1 on all coordinates $x_{i,j}$ except $x_{m,1}, \dots, x_{m,r}$. This coincides with the action of \tilde{G}_m above. The maps g_i are equivariant with respect to the above action, if one takes the trivial

action on \mathbb{R} . The fixed point sets $M^{\tilde{G}}, M_{m-1}^{\tilde{G}}$ coincide and are equal to the above defined C_m . Analogous to the \tilde{G}_i -action also the \tilde{G} -action is *CEF* on M_{i-1} for $i = 1, \dots, m$. One therefore gets inclusions

$$(3.5) \quad H_{\tilde{G}}^*(M) \xrightarrow{\gamma_1^*} H_{\tilde{G}}^*(M_1) \xrightarrow{\gamma_2^*} \dots \xrightarrow{\gamma_{m-1}^*} H_{\tilde{G}}^*(M_{m-1}) \xrightarrow{\gamma_m^*} H_{\tilde{G}}^*(C_m).$$

In order to calculate the maps γ_i^* above we also consider the Gysin map $\gamma_{i!}$ induced by the inclusion $\gamma_i : M_i \rightarrow M_{i-1}$. The composition $(\gamma_{i!})(\gamma_i^*)$ is given by the multiplication with the equivariant Euler class (cf. [16] or [5] for an account of equivariant Gysin homomorphisms, Euler classes, Thom classes etc.). In our case the Euler classes are given by the following Lemma.

Lemma 3.7. *For $i = 1, \dots, m-1$ the \tilde{G} -equivariant Euler class of the inclusion $\gamma_i : M_i \rightarrow M_{i-1}$ is $(t \otimes 1) \in H_{\tilde{G}}^*(M_{i-1}) \cong \mathbb{F}_2[t] \otimes H_G^*(M_{i-1})$.*

Proof. The Morse function $g_i : M_{i-1} \rightarrow \mathbb{R}$ is \tilde{G} -equivariant, where the \tilde{G} -action on \mathbb{R} is given by the trivial action of $G \subset \tilde{G}$, and the standard action of \mathbb{Z}_2 on \mathbb{R} . So the desired Euler class is just the pull back of the Euler class of the inclusion $\{0\} \subset \mathbb{R}$, which is $(t \otimes 1)$ as claimed. \square

Due to the above inclusions we view the elements in $H_{\tilde{G}}^*(M_i)$ also as elements in $H_{\tilde{G}}^*(M^{\tilde{G}}) = H_G^*(C_m)$ for $i = 0, \dots, m-1$. In particular s_J is divisible by t^{m-1} in $H_{\tilde{G}}^*(M^{\tilde{G}}) = H_G^*(C_m)$ (see 3.6). We use the following notation $s_J := \prod_{j \in J} s_j$ and $\bar{s}_J := \prod_{j \in J} \bar{s}_j$ for $J \subset \{1, \dots, r\}$ (cf. Proposition 3.5). Note that $l(J) := f_i(p_{i,J}) = \sum_{\{j \in J\}} l_j - \sum_{\{j \notin J\}} l_j$ is independent of i . (We point out that $l(J)$ coincides with L_J in [11], but not with $\ell(J)$ in [12].) In the literature the set J is called *short*, if

$$l(J) < 0, \text{ and } \textit{long}, \text{ if } l(J) > 0. \text{ For } i = 1, \dots, m \text{ we define } \mu_i(J) := \begin{cases} i & \text{for } l(J) > 0 \\ 0 & \text{for } l(J) < 0 \end{cases}$$

Let $\tilde{R} = H^*(\mathbb{Z}_2 \times (\mathbb{Z}_2)^r) = \mathbb{F}_2[t, t_1, \dots, t_r]$.

Theorem 3.8. *Let ℓ be generic. For $i = 0, \dots, m-1$ one has $H_{\tilde{G}}^*(M_i) \cong \tilde{R}\langle s_J/t^{\mu_i(J)}; J \subset \{1, \dots, r\} \rangle$, i.e. the \tilde{R} -subalgebra $H_{\tilde{G}}^*(M_i)$ of $H_{\tilde{G}}^*(M^{\tilde{G}})$ is generated by the elements $\{s_J/t^{\mu_i(J)}; J \subset \{1, \dots, r\}\}$ as a free \tilde{R} -module.*

We will give the proof of this theorem by alternating induction together with the proof of the following theorem.

Theorem 3.9. *Let ℓ be generic. For $i = 1, \dots, m$ one has $H_{\tilde{G}_i}^*(M_i) \cong \tilde{R}\langle s_J/t^{\mu_{i-1}(J)}; J \subset \{1, \dots, r\} \rangle / \tilde{R}\langle s_J/t^{i-1}(J), \bar{s}_J/t^{i-1}(J); l(J) > 0 \rangle$.*

Proof. The proof is by induction on i . The beginning, $i = 0$, is clear for Theorem 3.8 by Propositions 3.5. We proof Theorem 3.9 for i under the assumption that Theorem 3.8 holds for $i-1$. We apply Theorem 2.2 to the action of \tilde{G}_i and the decomposition $M_{i-1} = M_{i-1}^+ \cup_{M_{i-1}^0} M_{i-1}^-$ with $M_{i-1}^+ := \{w \in M_{i-1}; g_i(w) > 0\}$, $M_{i-1}^- := \{w \in M_{i-1}; g_i(w) < 0\}$ and $M_{i-1}^0 := \{w \in M_{i-1}; g_i(w) = 0\} = M_i$. Since l is generic, g_i does not vanish on C_i . So M_i is a compact manifold since the set of critical points of g_i is C_i ; and $(M_i)^{\tilde{G}_i}$ is empty. Using the fact that g_i is a perfect Morse function (see Proposition 3.4), and comparing the total dimensions of the (non-equivariant) cohomology modules of the respective ambient spaces and their fixed point sets, one sees, by Smith theory, that not only M_{i-1} but also M_{i-1}^\pm are

CEF with respect to \tilde{G}_i . So the assumptions of Theorem 2.2 are fulfilled. This gives, with the notation corresponding to that in Theorem 2.2,

$$(3.6) \quad H_{\tilde{G}_i}^*(M_i) \cong H_{\tilde{G}_i}^*(M)/(ker\xi^+ \oplus ker\xi^-)$$

Since the M_{i-1}^\pm are CEF , the kernels $ker\xi^\pm$ coincide with the kernels of the composition

$$(3.7) \quad H_{\tilde{G}_i}^*(M_{i-1}) \longrightarrow H_{\tilde{G}_1}^*(M_{i-1}^\pm) \longrightarrow H_{\tilde{G}_i}^*(C_i \cap M_{i-1}^\pm)$$

and Proposition 3.6 can therefore be used to identify these kernels. It follows from Proposition 3.6 that under the map induced by the inclusion $p_{i,J} \in M_{i-1}$ the element $s_I/t^{\mu_{i-1}(I)}$ is mapped to zero if and only if $I \not\subset J$ and the element $\bar{s}_I/t^{\mu_{i-1}(I)}$ is mapped to zero if and only if $I \cap J \neq \emptyset$. Since we have that all l_j are positive, $I \subset J$ implies $l(I) < l(J)$. One therefore gets that $ker\xi^\pm$ are generated as free $H_{\tilde{G}_i}^*(B\tilde{G}_i)$ -modules by $\{\bar{s}_J/t^{i-1}; l(J) > 0\}$ and $\{s_J/t^{i-1}; l(J) > 0\}$ respectively. Hence, by Theorem 2.2 and Proposition 3.5, we have

$$H_{\tilde{G}_i}^*(M_i) \cong \tilde{R}\langle s_J/t^{\mu_{i-1}(J)}; J \subset \{1, \dots, r\} \rangle / \tilde{R}\langle s_J/t^{i-1}, \bar{s}_J/t^{i-1}; l(J) > 0 \rangle.$$

So Theorem 3.9 holds for i .

We next show that Theorem 3.9 for i , (and Theorem 3.8 for $i-1$) imply Theorem 3.8 for i if $i < m$. We consider the following diagram

$$(3.8) \quad \begin{array}{ccc} H_G^*(M_{i-1}) & \longrightarrow & H_G^*(M_i) \\ \downarrow & & \downarrow \\ H_G^*(M_{i-1}) & \longrightarrow & H_G^*(M_i) \\ \uparrow & & \uparrow \\ H_{\tilde{G}_i}^*(M_{i-1}) & \longrightarrow & H_{\tilde{G}_i}^*(M_i) \end{array}$$

The horizontal maps are induced by $\gamma_i : M_i \longrightarrow M_{i-1}$. The vertical maps, which can be viewed as 'evaluation at $t = 0$ ', are induced by the inclusions $G \subset \tilde{G}$ and $G \subset \tilde{G}_i$, respectively. Since $H_G^*(M_{i-1})$, $H_G^*(M_i)$ and $H_{\tilde{G}_i}^*(M_{i-1})$ are free modules over $\tilde{R} = \mathbb{F}_2[t, t_1, \dots, t_r]$, the 'evaluation at $t = 0$ ' is surjective, i.e.

$H_G^*(M_{i-1}) \xrightarrow{t=0} H_G^*(M_{i-1})/tH_G^*(M_{i-1}) = H_G^*(M_{i-1}) \otimes_{\tilde{R}} R = H_G^*(M_{i-1})$, etc. for these modules, while $H_{\tilde{G}_i}^*(M_i) \xrightarrow{t=0} H_G^*(M_i)$ factors as

$H_{\tilde{G}_i}^*(M_i) \xrightarrow{t=0} H_{\tilde{G}_i}^*(M_i)/tH_{\tilde{G}_i}^*(M_i) \longrightarrow H_G^*(M_i)$. Note that $H_G^*(M_\mu)$ and $H_{\tilde{G}_i}^*(M_\mu)$ are isomorphic for $\mu < i$ since exchanging the coordinates $x_{i,j}$ and $x_{m,j}$ gives an equivariant homeomorphism of the \tilde{G}_i -space M_μ and the \tilde{G} -space M_μ . But this does not hold for $\mu = i$. In the above diagram elements of the form $(s_J/t^{i-1}; l(J) > 0) \in H_G^*(M_{i-1})$ are mapped to zero as one goes to $H_G^*(M_i)$, since the corresponding elements in $H_{\tilde{G}_i}^*(M_{i-1})$ go to zero by the above computation of $H_{\tilde{G}_i}^*(M_i)$. This implies that s_J/t^{i-1} can be divided by t in $H_G^*(M_i)$. Since multiplication with t is injective in $H_G^*(M_i)$ we get a uniquely determined element s_J/t^i for all J with $l(J) > 0$. On the other hand, from the above computation of $H_{\tilde{G}_i}^*(M_i)$ one gets that $H_{\tilde{G}_i}^*(M_i)_{t=0} = H_G^*(M_i)/tH_G^*(M_i)$ is a free R -module generated by the elements $\{s_I; l(I) < 0\}$, since $s_J \equiv 0 \equiv \bar{s}_J$ in $H_G^*(M_i)/tH_G^*(M_i)$ for $l(J) > 0$. (Note that $\bar{s}_J/t^{i-1} - s_J/t^{i-1}$ is divisible by t in $H_{\tilde{G}_i}^*(M_i)$, since $i < m$.) Hence the images of the

elements $\{s_I; l(I) < 0\}$ from $H_G^*(M_i)$ are linearly independent over $H^*(BG) = R$ in $H_{G_i}^*(M_i)_{t=0} \subset H_G^*(M_i)$.

We claim that the elements $\{s_I; l(I) < 0\} \sqcup \{s_J/t^i; l(J) > 0\}$ (in other words $\{s_J/t^{\mu_i(K)}; K \subset \{1, \dots, r\}\}$) freely generate $H_G^*(M_i)$ as $H^*(B\tilde{G})$ -module. Let $x \in H_G^*(M_i)$, then $\gamma_{i!}(x)$ can be written as

$$(3.9) \quad \gamma_{i!}(x) = \sum_{\{I; l(I) < 0\}} \lambda_I s_I + \sum_{\{J; l(J) > 0\}} \lambda_J s_J / t^{i-1}$$

with $\lambda_I, \lambda_J \in \tilde{R} = H^*(B\tilde{G})$.

So $tx = \gamma_i^* \gamma_{i!}(x) = \sum \lambda_I (s_I) + \sum \lambda_J t s_J / t^i$ in $H_G^*(M_i)$.

(Here we use the notation $\gamma_i^*(s_K) = s_K$ as elements in $H_G^*(M^{\tilde{G}})$.)

Evaluating at $t = 0$ gives $\sum \lambda_I \gamma_i^*(s_I) \equiv 0 \pmod{t}$ in $H_G^*(M_i)$.

Since $\{\gamma_i^*(s_I)\}$ are linearly independent in $H_G^*(M_i)$ over R , each coefficient $\lambda_I \in \tilde{R}$ must be divisible by t , i.e. $\lambda_I = t\xi_I$ in \tilde{R} , and hence $tx = \sum t\xi_I (s_I) + \sum t\lambda_J s_J / t^i$. Since division by t is unique in $H_G^*(M_i)$, we get $x = \sum \xi_I s_I + \sum \lambda_J s_J / t^i$. That means that $\{s_J/t^{\mu_i(K)}; K \subset \{1, \dots, r\}\}$ generate the subalgebra $H_G^*(M_i)$ of $H_G^*(M^{\tilde{G}})$, and these elements are linearly independent over $\tilde{R} = H^*(B\tilde{G})$, since this is clear after localization. \square

In a sense the (big) polygon spaces can be considered as a special case of the (big) chain spaces, if one allows the constant c to also take the value 0 in the definition of the chain spaces. We have shown that the assumption “ ℓ generic” implies, in particular, that $0 \in \mathbb{R}$ is a regular value of $g_m : M_{m-1} \rightarrow \mathbb{R}$. Actually for any regular value $c \in \mathbb{R}$ of g_m we can apply the last step of the proof of Theorem 3.9 to get the following generalization. Note that - due to the equivariant version of the Ehresmann fibration theorem - the equivariant diffeomorphism type of N_c does not change as c moves in an interval of only regular values of g_m (cf. [12]).

Theorem 3.10. *Let $\ell = (l_1, \dots, l_r)$ and $\tilde{\ell} = (l_1, \dots, l_r, c)$ be generic. Then*

$$(3.10) \quad H_G^*(N_c) \cong \tilde{R}\langle s_K / t^{\mu_{m-1}(K)}; K \subset \{1, \dots, r\} \rangle / (S + \overline{S})$$

with $S := \tilde{R}\langle s_I / t^{\mu_{m-1}(I)}; l(I) > -c \rangle$, and $\overline{S} := \tilde{R}\langle \overline{s}_J / t^{\mu_{m-1}(J)}; l(J) > c \rangle$.

We finally are interested in the equivariant cohomology of N_c with respect to the action of the subgroup $G \subset \tilde{G}$. It can be obtained as the middle term in a short exact universal coefficient sequence.

Proposition 3.11. *The following sequence is exact and splits.*

$$(3.11) \quad 0 \longrightarrow H_G^*(N_c) \otimes_{\tilde{R}} R \longrightarrow H_G^*(N_c) \longrightarrow \text{Tor}_{\tilde{R}}^1(H_G^*(N_c), R) \longrightarrow 0$$

The above Proposition follows from the next Lemma, which is probably well known. Since we could not find a reference in the literature for the splitting of the short exact sequence in the case at hand, we will provide a proof here.

Lemma 3.12. *Let*

$$(3.12) \quad 0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow C \longrightarrow 0$$

be an exact sequence of free differential graded R -modules with $H_*(A)$ and $H_*(B)$ free over R . Then the exact sequence

$$(3.13) \quad 0 \longrightarrow \operatorname{coker} \alpha_* \longrightarrow H_*(C) \longrightarrow \ker \alpha_* \longrightarrow 0$$

splits.

Proof. One has a short exact sequence

$$(3.14) \quad 0 \longrightarrow \operatorname{Hom}(C, D) \longrightarrow \operatorname{Hom}(B, D) \longrightarrow \operatorname{Hom}(A, D) \longrightarrow 0$$

where $D := \operatorname{coker} \alpha_*$, and a corresponding long exact sequence

$$(3.15) \quad \dots \rightarrow H^*(C; D) \rightarrow H^*(B; D) \rightarrow H^*(A; D) \rightarrow \dots$$

Since $H_*(A)$ and $H_*(B)$ are free, we have $H^*(A; D) \cong \operatorname{Hom}(H_*(A), D)$ and $H^*(B; D) \cong \operatorname{Hom}(H_*(B), D)$. The map $H^*(C; D) \rightarrow H^*(B; D)$ is the composition of the surjection $H^*(C; D) \rightarrow \operatorname{Hom}(D, D)$ and the injection $\operatorname{Hom}(D, D) \rightarrow \operatorname{Hom}(H_*(B), D)$. The first map factors through $\operatorname{Hom}(H_*(C), D)$. Hence $\operatorname{Hom}(H_*(C), D) \rightarrow \operatorname{Hom}(D, D)$ is also surjective and therefore $D = \operatorname{coker} \alpha_* \rightarrow H_*(C)$ has a splitting. \square

Proof. Proof of Proposition 3.11: Let

$$(3.16) \quad 0 \rightarrow \tilde{A} \xrightarrow{\tilde{\alpha}} \tilde{B} \rightarrow \tilde{C} \rightarrow 0$$

be a short exact sequence of free dg \tilde{R} -modules, which gives the sequence (3.18) in homology. Then the sequence

$$(3.17) \quad 0 \rightarrow A \xrightarrow{\alpha} B \rightarrow C \rightarrow 0$$

obtained from the above sequence by tensoring with R over \tilde{R} , fulfills the hypothesis of Lemma 3.12 since $H_*(\tilde{A}) = H_G^*(M_{m-1})$ and $H_*(\tilde{B}) = H_G^*(M_{m-1})/S \oplus H_G^*(M_{m-1})/\bar{S}$ are free over \tilde{R} , and hence $H_*(A) = H_G^*(M_{m-1})$ and $H_*(B) = H_G^*(M_{m-1})/S \oplus H_G^*(M_{m-1})/\bar{S}$ are free over R . Therefore, by Lemma 3.12, one has a short exact sequence

$$0 \rightarrow \operatorname{coker} \alpha_* \rightarrow H_G^*(N_c) \rightarrow \ker \alpha_* \rightarrow 0$$

which splits; with $\operatorname{coker} \alpha_* \cong H_G^*(N_c) \otimes_{\tilde{R}} R$ and $\ker \alpha_* \cong \operatorname{Tor}_{\tilde{R}}^1(H_G^*(N_c), R)$. \square

For $i = m$ we can also express the result of Theorem 3.9 in form of the short exact sequences, which are free resolutions of the \tilde{R} -module $H_G^*(N_c)$ (cf. (2.4) or (2.8))

$$(3.18) \quad 0 \rightarrow H_G^*(M_{m-1}) \xrightarrow{\tilde{\alpha}_*} H_G^*(M_{m-1})/S \oplus H_G^*(M_{m-1})/\bar{S} \rightarrow H_G^*(N_c) \rightarrow 0$$

or

$$(3.19) \quad 0 \rightarrow \bar{S} \xrightarrow{\tilde{\iota}} H_G^*(M_{m-1})/S \rightarrow H_G^*(N_c) \rightarrow 0$$

with S and \bar{S} as above.

Therefore we can compute the tensor and tor term in (3.11) by just taking the cokernel and the kernel of the map $\tilde{\alpha}_*$ evaluated at $t = 0$, i.e. of $\alpha := \tilde{\alpha}_* \otimes_{\tilde{R}} R$ or of $\iota := \tilde{\iota} \otimes_{\tilde{R}} R$.

We have the following exact sequence

$$(3.20) \quad 0 \rightarrow \ker \iota \rightarrow \bar{S} \otimes_{\tilde{R}} R \xrightarrow{\iota} (H_G^*(M_{m-1})/S) \otimes_{\tilde{R}} R \rightarrow \operatorname{coker} \iota \rightarrow 0$$

where $\bar{S} \otimes_{\bar{R}} R$ and $(H_G^*(M_{m-1})/S) \otimes_{\bar{R}} R$ are free R -modules of rank $|\{J; l(J) > c\}|$ and $|\{I; l(I) < -c\}| = |\{J; l(J) > c\}|$ respectively. We want to compute the map ι . Since $\bar{s}_j = s_j + t^{m-1}t_j^n$ one has $\bar{s}_J = s_J + \sum_{j \in J} t^{m-1}t_j^n s_{(J \setminus \{j\})} + t^{2(m-1)}(\dots)$, so

$$(3.21) \quad \tilde{\iota}(\bar{s}_J) \equiv \sum_{\{j \in J; l(J \setminus \{j\}) < -c\}} t^{m-1}t_j^n s_{(J \setminus \{j\})} + t^{2(m-1)}(\dots)$$

in $H_G^*(M_{m-1})/S$ for $l(J) > c > 0$, and

$$(3.22) \quad \iota(\bar{s}_J/t^{m-1}) \equiv \sum_{\{j \in J; l(J \setminus \{j\}) < -c\}} t_j^n s_{(J \setminus \{j\})}$$

in $(H_G^*(M_{m-1})/S) \otimes_{\bar{R}} R$.

Corollary 3.13. *For $n > 0$ the equivariant cohomology $H_G^*(N_c)$ is a free R -module if $c \geq l_j$ for $j = 1, \dots, r$, i.e. if (l_1, \dots, l_r, c) is a dominated length vector in the sense of [9]. On the other hand $H_G^*(N_0)$ is never free (cf. [12]), Lemma 4.3).*

Proof. We use the criterion, established by Smith theory, that $H_G^*(N_c)$ is free if and only if $\dim_{\mathbb{F}_2} H^*(N_c) = \dim_{\mathbb{F}_2} H^*(N_c^G)$. Assume that $c = 0$. Since $n > 0$ we can rename the variables $y_{n,1}, \dots, y_{n,r}$ as $x_{m+1,1}, \dots, x_{m+1,r}$, thereby replacing m by $m+1$, and n by $n-1$. We can now apply Cor. 3.4 in the new setting to obtain that $\dim_{\mathbb{F}_2} H^*(M_m) = 2^r$ (cf. [12], Prop. 3.3). On the other hand one can also use Theorem 3.8, Theorem 3.9 and Proposition 3.11 to compute $H_{\mathbb{Z}_2}^*(N_0^G)$ and $H^*(N_0^G)$. This corresponds to the case $n = 0$. One obtains from the sequence (3.20) that $\dim_{\mathbb{F}_2} H^*(N_0^G) = 2^r - 2rk(\iota) < 2^r$, because there always exists a J with $l(J) > 0$ such that $l(J \setminus \{j\}) < 0$ for some $j \in J$. Therefore ι is not trivial. Hence $H_G^*(N_0)$ is never free. Also for $c \neq 0$ one can use the above calculation to get $\dim_{\mathbb{F}_2} H^*(N_c^G) = 2|\{J; l(J) > c\}| - 2rk(\iota)$, but ι is trivial here if $c \geq l_j$ for $j = 1, \dots, r$. To calculate $H^*(N_c)$ we rename the variables $(y_{\nu,1}, \dots, y_{\nu,r})$ as $x_{m+1,1}, \dots, x_{m+1,r}$ for $\nu = 1, \dots, n$ thus replacing m by $m+n$ and n by 0. The group G in this new setting is just $\{1\}$ and $\tilde{G} = \mathbb{Z}_2$. So $\bar{s}_j = s_j + t^{m+n-1}$ in this setting. But the number of hyperplanes intersecting the product of spheres is m . And therefore the appropriately modified Remark 3.10 gives

$$(3.23) \quad H_{\mathbb{Z}_2}^*(N_c) \cong \tilde{R}\langle s_K/t^{\mu_{m-1}(K)}; K \subset \{1, \dots, r\} \rangle / (S + \bar{S})$$

with $S := \tilde{R}\langle s_I/t^{\mu_{m-1}(I)}; l(I) > -c \rangle$, and $\bar{S} := \tilde{R}\langle \bar{s}_J/t^{\mu_{m-1}(J)}; l(J) > c \rangle$. But ι turns out to be trivial in this case since $\bar{s}_J/t^{m-1} - s_J/t^{m-1} \equiv 0 \mod t$. So $\dim_{\mathbb{F}_2} H^*(N_c) = \dim_{\mathbb{F}_2} \ker \iota + \dim_{\mathbb{F}_2} \operatorname{coker} \iota = 2|\{J; l(J) > c\}|$ (see (3.20)). \square

Examples below show (see Example 3.19 (3)), that $H_G^*(N_c)$ can be free for $c \neq 0$ even if (l_1, \dots, l_r, c) is not a dominated length vector. Due to the equivariant version of the Ehresmann fibration theorem varying the constant c between two adjacent critical values of g_m does not change the equivariant diffeomorphism type of N_c . So $N_c \cong N_0$ if $0 < c < cr_{\min}$ where cr_{\min} denotes the minimal positive critical value of g_m . We extend the analogue of [12], Cor.6.4 to the situation where c is not necessarily equal to 0. Two length vectors are considered equivalent, if they induce the same notion of 'long' and 'short' index sets.

Proposition 3.14. (1) *If $r = 2k + 1$ then $H_G^*(N_c)$ has syzygy order k if and only if (l_1, \dots, l_r) is equivalent to $(1, \dots, 1)$ and $0 < c < cr_{\min}$.*

(2) *If $r = (2k + 2)$ then $H_G^*(N_c)$ has syzygy order k if and only if (l_1, \dots, l_r) is equivalent to $(0, 1, \dots, 1)$ and $0 < c < cr_{\min}$.*

Proof. The proof is a modification of the proof of [12], Prop.6.4.

Let $\mathcal{L}_c = \{J; l(J) > c\}$ and $\mathcal{S}_c = \{J; l(J) < -c\}$. The map which assigns to a subset $J \subset \{1, \dots, r\}$ its complement \bar{J} gives a bijection between \mathcal{L}_c and \mathcal{S}_c . Let \mathcal{L}'_c be the subset of $\mathcal{L}_c = \{J; l(J) > c\}$, such that there exists an $j \in J$ with $(J \setminus \{j\}) \in \mathcal{S}_c = \{J; l(J) < -c\}$.

(1) Assume $r = 2k + 1$ and that the syzygy order of $H_G^*(N_c)$ is k . Since $H_G^*(N_c)$ is not free, \mathcal{L}'_c can not be empty; and since the syzygy order is k , any index set $J \in \mathcal{L}'_c$ contains at least $k + 1$ indices j , such that $(J \setminus \{j\}) \in \mathcal{S}_c$ (cf. [12], Prop.6.3). In particular any set $J \in \mathcal{L}'_c$ must have at least $k + 1$ elements. The complement \bar{J} of J has at most m elements and lies in \mathcal{S}_c . But $\bar{J} \cup \{j\}$ is in \mathcal{L}'_c , since its complement $J \setminus \{j\}$ is in \mathcal{S}_c . So $\bar{J} \cup \{j\}$ and also J must have precisely $k + 1$ elements for a sets $J \in \mathcal{L}'_c$. Removing an element j from a set J as above and replacing it by an element i from \bar{J} gives again a set in \mathcal{L}'_c . It therefore follows, that the sets in \mathcal{L}'_c are all those having precisely $k + 1$ elements and the sets in \mathcal{L}_c all those which have at least $k + 1$ elements. This means that sets with less than $k + 1$ elements are in \mathcal{S}_c . All together \mathcal{L}_c , resp. \mathcal{S}_c , coincide with the long, resp. short, subsets for the length vector $(1, \dots, 1)$, and $c < cr_{min}$ since there must be an index set K with $l(K) = cr_{min}$. This proves part (1).

(2) We assume again that the syzygy order of $H_G^*(N_c)$ is k , but this time $r = 2k + 2$. We define \mathcal{L}_c and \mathcal{S}_c and \mathcal{L}'_c as before. We may assume that $l_1 < l_2 \leq l_3 \dots \leq l_r$ because the equivariant diffeomorphism type of N_c does not change under small enough perturbations of the (l_1, \dots, l_r) and of c . Again \mathcal{L}'_c is not empty, and for any set $J \in \mathcal{L}'_c$, one has $J \setminus \{j\} \in \mathcal{S}_c$ for at least $k + 1$ elements in J . Arguing similarly to case (1) one gets that either both J and \bar{J} have precisely $k + 1$ elements or J has $k + 2$ elements and therefore \bar{J} has k elements. If $J \in \mathcal{L}'_c$ then $\bar{J} \setminus \{j\} = \bar{J} \cup j$ is also in \mathcal{L}'_c for at least $k + 1$ elements $j \in J$. So \mathcal{L}'_c must contain index sets with $k + 2$ elements and also index sets with $k + 1$ elements. Assume $J \in \mathcal{L}'_c$ contains 1 and has precisely $k + 1$ elements. Then $\bar{J} \setminus \{j\} = \bar{J} \cup j$ is also in \mathcal{L}'_c for all elements $j \in J$. Also $I := \bar{J} \setminus \{j\} = \bar{J} \cup j$ contains at least $m + 1$ elements i , such that $\bar{I} \setminus \{i\} = J \setminus \{j\} \cup \{i\}$ is also in \mathcal{L}'_c . Listing the indices occurring in J , resp. \bar{J} weakly increasing one gets two sequences $1, a_2, \dots, a_{k+1}$ resp. b_1, \dots, b_{k+1} . The above argument shows that one can replace a_ν by b_ν if $a_\nu > b_\nu$ for $\nu = 2, \dots, m + 1$. For the index sets, K and \bar{K} , obtained this way one still has $1 \in K$ and $K \in \mathcal{L}'$. But $l(K) < l(\bar{K})$, which is impossible, since $l(K) > c$. So there can't be index sets $J \in \mathcal{L}'_c$ containing 1 and precisely $k + 1$ elements. Assume $J \in \mathcal{L}'_c$ contains 1 and has $k + 2$ elements. Similar to the above reasoning one sees that replacing an index $j \neq 1$ in J by an index i in \bar{J} gives again an index set in \mathcal{L}'_c . It follows that all index sets, which contain 1 and have precisely $k + 2$ elements are in \mathcal{L}'_c . So \mathcal{L}_c , resp. \mathcal{S}_c , coincide with the long, resp. short, subsets for the length vector $(0, 1, \dots, 1)$. As above one sees that $c < cr_{min}$. \square

Remark 3.15. (1) We would like to point out that Theorem 3.8 and Theorem 3.9 give complete information about the product structure of the equivariant cohomology with respect to the \tilde{G}_i -actions, but Proposition 3.11 gives only partial information about the product in $H_G^*(N_c)$.

(2) Theorem 3.8, Theorem 3.9 and Proposition 3.11 can be applied if $n = 0$ and $G = \{1\}$ (see e.g. the proof of Cor. 3.13). As a special case one obtains the \mathbb{Z}_2 -equivariant and the non-equivariant cohomology of the spaces in Remark 3.1,(2)

-(4), which are studied in several papers, see e.g. [10], [9], [11], [15]. In case $m > 2, n = 0$ and $c = 0$ the term $H_G^*(N_0) \otimes_{\tilde{R}} R = H_{\mathbb{Z}_2}^*(E_m(\ell)) \otimes_{\mathbb{F}_2[t]} \mathbb{F}_2$, including the multiplicative structure, is just $H^{(m-1)*}(E_m(\ell); \mathbb{F}_2)$ studied in detail in [11], Section 4. One gets that $\dim_{\mathbb{F}_2} H^*(E_m(\ell))$ is smaller than 2^r . A sharp upper bound is contained in [10], Thm. 2].

(3) Since, for $i = 1, \dots, m-1$, the spaces M_i are CEF with respect to the \tilde{G} -action, one obtains the equivariant cohomology with respect to subgroups $G' \subset \tilde{G}$, and in particular the non-equivariant cohomology (for the trivial subgroup $\{1\}$), just as the tensor product $H_{\tilde{G}}^*(M_i) \otimes_{H^*(B\tilde{G})} H^*(BG')$.

While $H_{G'}^*(M_i)$ is free over $H^*(BG')$ for any $G' \subset \tilde{G}_i$ and $i = 1, \dots, m-1$, the equivariant cohomology $H_{\tilde{G}}^*(M_m)$ is torsion, which already follows from the fact that $(M_m)^{\tilde{G}} = \emptyset$. As mentioned above the equivariant cohomology $H_G^*(M_m)$, $(M_m = N_0)$, with respect to G is not free, but it is often torsion-free over R (see [12], Section 5 and 6). We will not perform explicit calculations here for the general case. For the "big polygon spaces" (and the corresponding $(S^1)^r$ -action in the complex situation) these are done and discussed in [12]. This could be imitated in the real case at hand in a similar vein. But the following example (cf. [12], Prop. 5.1) gives - from the view point of syzygies - perhaps the most interesting special case and already shows some typical features of the general case for the "big polygon spaces". Later on we discuss some examples of "big chain spaces".

Example 3.16. We assume $m = 2, n = 1, r = 2k + 1, k \geq 0$ and $\ell := (1, \dots, 1)$. Under this assumptions one has $l(J) > 0$ if and only if $|J| > k$. The map ι defined above turns out in this case to be trivial on \bar{s}_J/t for $l(J) > k + 1$. For $l(J) = k + 1$ it coincides with the following boundary map in the Koszul complex $\delta : \Lambda_R^{k+1}(\sigma_1, \dots, \sigma_r) \rightarrow \Lambda_R^k(\sigma_1, \dots, \sigma_r)$ if one puts $R\langle \bar{s}_J/t; l(J) = k + 1 \rangle \cong \Lambda_R^{k+1}(\sigma_1, \dots, \sigma_r)$ and $R\langle s_I; l(I) = k \rangle \cong \Lambda_R^k(\sigma_1, \dots, \sigma_r)$. We therefore get

$$(3.24) \quad \text{coker } \iota \cong R\langle s_I, |I| < k \rangle \oplus \text{coker } (\delta : \Lambda_R^{k+1}(\sigma_1, \dots, \sigma_r) \rightarrow \Lambda_R^k(\sigma_1, \dots, \sigma_r))$$

and

$$(3.25) \quad \ker \iota \cong R\langle s_J/t, |J| > k + 1 \rangle \oplus \ker (\delta : \Lambda_R^{k+1}(\sigma_1, \dots, \sigma_r) \rightarrow \Lambda_R^k(\sigma_1, \dots, \sigma_r))$$

Finally, from the short exact Künneth sequence, which splits, we get $H_G^*(N_0) \cong \text{coker } \iota \oplus \ker \iota$ as R -modules. In particular, since $\text{coker } (\delta : \Lambda_R^{k+1} \rightarrow \Lambda_R^k)$ is a k -th, but not a $(k+1)$ -th syzygy and $\ker (\delta : \Lambda_R^{k+1} \rightarrow \Lambda_R^k)$ is a $(k+2)$ -th syzygy, we get the following result.

For Example 3.16 one has:

Corollary 3.17. *The equivariant cohomology $H_G^*(N_0)$ is a k -th syzygy, but not a $(k+1)$ -th syzygy.*

Remark 3.18. The above Corollary 3.17 shows that the maximal bound (namely k) for the syzygy order (in the non-free case) given by Proposition 2.1 for an action of $(\mathbb{Z}/2)^{2k+1}$ can be realized by the equivariant cohomology of a compact manifold. It is pointed out in [12], (5.2) that, using such "maximal" examples, one can easily realize all other orders of syzygies allowed by Proposition 2.1. One can just extend the action to a larger rank torus letting the extra coordinates act trivially. This obviously changes the rank of the torus, but it does not change the syzygy order of the equivariant cohomology. It corresponds to extending the length vector by

another coordinate equal to 0, and it is easy to check that this does not change the syzygy order but increases the rank of the torus acting. In [12] there is a careful discussion of the effect of different length vectors on the syzygy order in case of $(S^1)^r$ -actions. This could as well be imitated for the $(\mathbb{Z}_2)^r$ -manifolds considered here.

We finish with a few examples of "big chain spaces", a class of spaces which is not considered in [12]. They show that the syzygy order of $H_G^*(N_c)$ depends in a rather delicate way on the length vector ℓ and the constant c . But using Theorem 3.10 and Proposition 3.11 the calculation of the equivariant cohomology is straight foreword and we leave the details to the reader. Again one can change the rank of the torus acting without changing the syzygy order by adding coordinates equal to 0 to the length vector. In particular one obtains corresponding examples for tori of even rank this way.

Example 3.19. (1) Let $r = 2k + 1$ and $\ell = (1, \dots, 1)$. The critical values of g_m are $\{-r, -(r-2), \dots, -1, 1, \dots, (r-2), r\}$. Recall that for a regular value, c of g_m , one has $H_G^*(N_c) \cong H_G^*(N_{-c})$. Let $0 \leq c$ be a regular value of g_m , then

$$H_G^*(N_c) \begin{cases} \text{has syzygy order } k & \text{if } 0 \leq c < 1 \\ \text{is free} & \text{if } c > 1 \end{cases}$$

This does not mean that $H_G^*(N_c)$ is the same for all $c > 1$. The rank decreases as c increases, crossing critical values of g_m ; in particular $H_G^*(N_c) = 0$ if $c > r$.

(2) Let $r = 2k + 1$, $\ell = (2, 2, 3, \dots, 3)$ and c a regular value of g_m , then

$$H_G^*(N_c) \begin{cases} \text{has syzygy order } k & \text{if } 0 \leq c < 1 \\ \text{has syzygy order } (k-1) & \text{if } 1 < c < 3 \\ \text{is free of decreasing rank} & \text{if } c > 3 \end{cases}$$

(3) Let $\ell = (2, 2, 2, 3)$, then

$$H_G^*(N_c) \begin{cases} \text{has syzygy order } 0 & \text{if } 0 \leq c < 1 \\ \text{is free} & \text{if } c > 1 \end{cases}$$

Note that for the first two cases in Example 3.19 one gets the same result for $c = 0$, i.e. for the big polygon spaces, but not for all values of c , i.e. not for all big chain spaces. In case (3) one gets freeness of the equivariant cohomology even for constants c which do not dominate ℓ (cf. Cor. 3.13).

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